

Lagrange equations for the $\hat{k}$ -cross basis exponential (Lagrangian of time dependent harmonics)

Whilst the $\hat{k}$ basis exponential **expk** is a solution to the harmonic oscillator it represents only one of the roots of sine. The harmonic oscillator, expressed as a sine or a cosine, has two roots. The other root is the conjugate **expk**. So the complete solution to the harmonic oscillator is **expk** and **expk** ' (**conjugate expk**) taken together as some orthogonal linear combination. What motivated this eight page note is the desire to express coordinate systems and transformations between coordinate systems as a differential equation(s). I had first thought that the coordinate system transformation [perhaps slightly naively] was the harmonic oscillator [this is for light] but it is actually **expk** itself. Scaling the rectangular basis by **expk** = **cisk** results in the cylindrical basis system [for a rotating reference frame for example]. Note here that cylindrical and spherical geometry is ubiquitous in electromagnetism and these geometries are generated from the rectangular system by scaling with **expk**=**cisk**, **expj**=**cisk** and **expi**=**cisi** [note further that this is more suggestion that electromagnetism is a coordinate system of basis exponentials]. The idea of **cis-scaling a coordinate system** [which I have shifted this from, and have yet to finish] lends itself nicely to this topic of the **Lagrangian**, and hence the possibility of the Lagrangian of a coordinate system. In this instance, the Lagrangian of a rotating reference frame [cylindrical system]. The difference between the two solutions, that is whether **expk** and its conjugate (**expk** or **expk**') are the solutions to the coordinate system, is the same difference between that of the **characteristic polynomial** of **expk** (the $\hat{k}$ -hat exponential being the rotation matrix) and its **minimum polynomial**, the two of which (characteristic and minimum polynomials) frequently not coinciding as is the case for the rotation matrix in R^3 . Having just studied the mathematics of transmission lines (as part of 3rd year electromagnetism), the solution to the Telegrapher's equations (the two equations [one for current one for voltage] on which transmission line theory is based) is the ($\hat{k}$ -hat) basis exponential (**expk**), which is the resolved form of a single root ($\hat{k}$ -cross). The other root (in resolved form, the conjugate basis-exponential **expk**') is added to the solution based on arguments of reflection of the wave from the load, taken together then being the full solution. So not trivial.

The relevant vectors are the **forward diffusion (spatial) variable for position (R)**, the **forward diffusion variable for velocity (R-dot)**, and the corresponding **backward variables (r and r-dot)**. These four variables are respectively

$$\vec{R}_t = (X_t, Y_t, Z_t)' \quad \dot{\vec{R}}_t = (\dot{X}_t, \dot{Y}_t, \dot{Z}_t)' \quad \vec{r}_0 = (x_0, y_0, z_0)' \quad \dot{\vec{r}}_0 = (\dot{x}_0, \dot{y}_0, \dot{z}_0)'$$

The Lagrange equation for each R^3 component direction is

$$\frac{d}{dt} \frac{\partial X_t}{\partial \dot{x}_0} - \frac{\partial X_t}{\partial x_0} = 0 \quad \frac{d}{dt} \frac{\partial Y_t}{\partial \dot{y}_0} - \frac{\partial Y_t}{\partial y_0} = 0 \quad \frac{d}{dt} \frac{\partial Z_t}{\partial \dot{z}_0} - \frac{\partial Z_t}{\partial z_0} = 0$$

where the Lagrangian in this context is the components of the **forward diffusion (spatial) variable** in R^3 .

By the theory of $\hat{k}$ -cross, the **minimum polynomial** of the $\hat{k}$ -hat exponential [and **transfer function**] is $p(D) = D + \omega \hat{k} \times$ [I am writing a note after this on the similarity relations of the basis cross products and have a short proof of the last equation which I will include there]. This is the basis-cross factorised form of the rotation matrix (I was the first one to put two and two together on this one). So **chains of these roots** in a **transfer function** would then be **chains of basis exponentials** which is how I intend to get all of the sophistication I keep talking about with higher order differential equations... that is, with long chains of these [which will have a generalized long-chain Lagrangian which I will write up later (which allow for overtones in velocities and potentials ... which can extend theoretical physics)]. Note that this **minimum polynomial** is first order despite the harmonics in the rotation matrix being second, that is, a second order **characteristic equation** $\phi(D) = (D + e^{\omega \hat{k} \times})(D - e^{-\omega \hat{k} \times})$ which shows that characteristic equation [in this format] is second order whilst the minimum polynomial is first order [same sort of issue with the Telegrapher's equations]. I won't go into the detail here as I will be doing so in the documents on calculus proper as this is only a several page note on the Lagrangian of basis-exponentials. The unforced solution is the solution to $(D + \omega \hat{k} \times) \vec{R}_t = 0$. The unforced solution is

$$\vec{R}_t = e^{-\omega t \hat{k} \times} \vec{r}_0' \quad \text{i.e.,} \quad (X_t, Y_t, Z_t)' = e^{-\omega t \hat{k} \times} (x_0, y_0, z_0)' \quad \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = e^{-\omega t \hat{k} \times} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

which is the $\hat{k}$ -hat basis exponential and is the **null space**, that is, the $\hat{k}$ -hat basis exponential (the $\hat{k}$ -y plane rotation matrix) is the **kernel of the minimum polynomial** $p(D) = D + \omega \hat{k} \times$. Since it is diagonalizable, the **algebraic and geometric multiplicities** coincide. The algebraic and geometric multiplicities coincide since it is diagonalizable, and this arises in this circumstance because the rotation matrix satisfies de-Moivre's theorem which make the rotations continuous for each coordinate axis of the rotation plane. Hence three independent eigenvectors. Exponentiating $\hat{k}$ -cross essentially converts the **discrete quartic roots of unity** of $\hat{k}$ -cross into the continuous **infinite roots of unity** on the rotation plane [around the rotation axis] of **expk**. By Euler's formula (**expk** = **cisk**), the solution is

$$\vec{R}_t = \vec{r}_0 (\cos(\omega t) - \sin(\omega t) \hat{k} \times) \quad (X_t, Y_t, Z_t)' = (x_0, y_0, z_0)' (\cos(\omega t) - \sin(\omega t) \hat{k} \times) \quad \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} (\cos(\omega t) - \sin(\omega t) \hat{k} \times)$$

The distinction between vectors and scalars becomes a little blurred for the components of the spatial variables (X , Y , and Z that comprise R) since all variables are essentially vectors within R^3 . The question then is “does **one-to-one** and **onto**” [injective+surjective=bijjective] still have the same meaning or do these terms need updating or extension? This question is relevant for functions that map **from i -hat to j -hat**, say, or **from i -hat to both i -hat and j -hat** as per the real R^3 interpretation of the complex plane which is the plane of rotation in R^3 . I do not provide the answer to this here, only note that it is a question that will probably need to be answered at some stage perhaps with additional terminology.

For the variables of the plane X and Y , there is one real and one complex eigenvector, that is, a cosine to the assigned axis [which vector expands to the first cosine and sine column basis vector] and a sine mapping to an orthogonal axis [and hence vector expands to the second cosine and sine column basis vector]. For the k -hat rotation axis it is unity. The complex eigenvector corresponding to the sine can be made real [in the context of standard complex terminology] that is scalar on the same axis [in new vector basis-cross-product terminology] with a trick which then makes it possible to state the components in terms of the Lagrange equation which are ordinarily scalar [non-complex (non-vectorial)] equations. Note that the Lagrange equations have solved thousands of major physical and engineering problems. This will [possibly] make these equations even more useful.

So I now prove that the time dependent k -hat basis-exponential (time dependent complex exponential) satisfies Lagrangian theory, and does so in the real vector space R^3 . This is because a Lagrangian essentially fits a harmonic: the leading term fits the even cosine axis and the trailing term fits the odd sine axis, the odd sine axis being the derivative of the even axis where derivative in this context means a 90 degree phase shift (as for phasors). This can all be explained and interpreted physically with a pendulum where it is made clear that the harmonic is associated with oscillations between stored and kinetic energy. Doesn't really matter which but one of the axes is the stored energy the other kinetic and these two points are spanning (similarly as k -cross and conjugate) for parameterizing the system as per the Nyquist theorem, that is, two points is sufficient for any one period. In the context of a coordinate system, this is where the rectangular system is set rotating at a constant frequency ω . I am hoping that it will also be possible to do this for the **spherical system** [with a special interest in **Foucault's pendulum** and **Lissajous figures**, ultimately (or possibly rather), for the **precession of magnetic moments of nuclei** ... having problems with this actually because k -cross doesn't commute with the j -hat basis exponential **expj** (or **expi**, only with **expk** and **expkk**) and this poses difficulties with inverting $p(D) = D + \omega_y \hat{j} \times + \omega_z \hat{k} \times$].

There are two approaches I have considered for the Lagrangian, both give the same result for the components of the plane of rotation but give **conflicting results** for the **rotation axis**, that is, for the k -hat component. For each of the three components in R^3 of the position vector R , the **first approach** is to hold on taking directions whilst in the **second approach** directions are taken at the outset. The difference in outcome for the second approach is that the k -hat direction does not satisfy the Lagrange equation, instead it returns what I think is the rotation eigenvalue [with a sign change] of the rotation. Time will tell which is best. Now after a few months consideration, I am pretty sure the second approach is the correct one. I have not included the working for the first approach as I think it is incorrect. Below, then, is the maths of the second approach. The differential equations [transfer functions] of each of the three components are

$$DX_t + \omega X_t \hat{k} \times = 0$$

$$DY_t + \omega Y_t \hat{k} \times = 0$$

$$DZ_t + \omega Z_t \hat{k} \times = 0$$

For the second approach, taking directions immediately

in the **i -hat direction**

in the **j -hat direction**

in the **k -hat direction**

$$(D + \omega \hat{k} \times) X_t \hat{i} = 0$$

$$(D + \omega \hat{k} \times) Y_t \hat{j} = 0$$

$$(D + \omega \hat{k} \times) Z_t \hat{k} = 0$$

$$DX_t \hat{i} + \omega X_t \hat{k} \times \hat{i} = 0$$

$$DY_t \hat{j} + \omega Y_t \hat{k} \times \hat{j} = 0$$

$$DZ_t \hat{k} + \omega Z_t \hat{k} \times \hat{k} = 0$$

$$\hat{i} DX_t + \hat{j} \omega X_t = 0$$

$$\hat{j} DY_t - \hat{i} \omega Y_t = 0$$

$$\hat{k} DZ_t = 0$$

$$\hat{i} \dot{x}_0 + \hat{j} \omega x_0 = 0$$

$$\hat{j} \dot{y}_0 - \hat{i} \omega y_0 = 0$$

$$\hat{k} \dot{z}_0 = 0$$

where the trick here is with the **initial condition** which converts the complex part into a real initial derivative. If the trick is right, the last line says that the initial conditions, position and derivative, are out of phase by 90 degrees and scale by ω , and hence noting for the lines of working for the tentative substitutions $\hat{j} x_0 = \hat{i} (-\dot{x}_0 / \omega)$ and $\hat{i} y_0 = \hat{j} (\dot{y}_0 / \omega)$ which I am now confident of (although wasn't initially since this is new mathematics and the definitions of **one-to-one** and **onto** for mappings between components does not yet exist).

This is [was] tentative until I could answer the following question in the affirmative: does it makes sense that $\hat{j} x_0$ and $\hat{i} (-\dot{x}_0 / \omega)$ are equal quantities and so can be treated as algebraic quantities for algebraic manipulation in equations? This seems inconsistent with equating real and imaginary parts which would suggest that these quantities are zero which I am pretty confident are not. Similarly

with equating $\hat{i} y_0$ and $\hat{j}(\dot{y}_0 / \omega)$ for the j -hat component for Y (although the k -hat component for Z is certainly zero for $\dot{z}_0$). If it is possible to say that, say, $\hat{i} a$ and $\hat{j} b$ are equal so can be equated and treated as scalars [that is, manipulated algebraically as $x = (\hat{i} a)$ and $y = (\hat{j} b)$], this would add a whole new dimension to the scope of algebra With quite a bit of further consideration, and my absolute belief in the goodness of complex analysis in which one definitely equates real and complex parts independently, independently of whether it does or doesn't for any arbitrary a and b coupled with directions, it does certainly in this instance because one is the time-dot of the other and since this is calculus, in particular the calculus of a basis exponential, the derivative of the initial condition (the initial velocity) essentially acts as a substitute for k -cross... so yeah, it works. In their respective directions, with the substitutions $\hat{j} x_0 = \hat{i}(-\dot{x}_0 / \omega)$ and $\hat{i} y_0 = \hat{j}(\dot{y}_0 / \omega)$,

 i -hat component (rotation plane)

$$\hat{i} \dot{x}_0 = \hat{j}(-\omega x_0) \Leftrightarrow \hat{j} x_0 = \hat{i}(-\dot{x}_0 / \omega)$$

 j -hat component (rotation plane)

$$\hat{j} \dot{y}_0 = \hat{i}(\omega y_0) \Leftrightarrow \hat{i} y_0 = \hat{j}(\dot{y}_0 / \omega)$$

 k -hat component (rotation axis)

$$\hat{k} \dot{z}_0 = 0$$

the component solutions are:

 i -hat component (rotation plane)

$$\begin{aligned} X_t \hat{i} &= x_0 e^{-\omega t \hat{k} \times \hat{i}} \\ &= x_0 (\cos \omega t - (\sin \omega t) \hat{k} \times \hat{i}) \\ &= (\hat{i} x_0 \cos \omega t - \hat{j} x_0 \sin \omega t) \\ &= (\hat{i} x_0 \cos \omega t - \hat{i}(-\dot{x}_0 / \omega) \sin \omega t) \\ &= \hat{i} (x_0 \cos \omega t + (\dot{x}_0 / \omega) \sin \omega t) \end{aligned}$$

 j -hat component (rotation plane)

$$\begin{aligned} Y_t \hat{j} &= y_0 e^{-\omega t \hat{k} \times \hat{j}} \\ &= y_0 (\cos \omega t - (\sin \omega t) \hat{k} \times \hat{j}) \\ &= (\hat{j} y_0 \cos \omega t + \hat{i} y_0 \sin \omega t) \\ &= (\hat{j} y_0 \cos \omega t + \hat{j}(\dot{y}_0 / \omega) \sin \omega t) \\ &= \hat{j} (y_0 \cos \omega t + (\dot{y}_0 / \omega) \sin \omega t) \end{aligned}$$

 k -hat component (rotation axis)

$$\begin{aligned} Z_t \hat{k} &= z_0 e^{-\omega t \hat{k} \times \hat{k}} \\ &= z_0 (\cos \omega t - (\sin \omega t) \hat{k} \times \hat{k}) \\ &= (\hat{k} z_0) \\ &= \hat{k} z_0 \end{aligned}$$

which are hence now scalar equations for each component, i.e., with $\hat{j} x_0 = \hat{i}(-\dot{x}_0 / \omega)$ and $\hat{i} y_0 = \hat{j}(\dot{y}_0 / \omega)$ the vector components of the basis exponential are converted to scalar equations, and now have all of the independent variables required for the Lagrange equation, notably the **backward variables** (x_0 and $\dot{x}_0$) and **time variable** (t):

 i -hat component (rotation plane)

$$X_t = x_0 \cos \omega t + (\dot{x}_0 / \omega) \sin \omega t$$

 j -hat component (rotation plane)

$$Y_t = y_0 \cos \omega t + (\dot{y}_0 / \omega) \sin \omega t$$

 k -hat component (rotation axis)

$$Z_t = z_0$$

and in the form which satisfy Lagrange equation, with the backward diffusion (spatial) variable x_0 , the backward velocity $\dot{x}_0$, and the time variable t for forward transition. Now demonstrating that the components of the k -hat basis exponential satisfy Lagrange's equation for the plane of rotation

 i -hat component (rotation plane)

$$\begin{aligned} \frac{\partial X_t}{\partial x_0} &= \cos \omega t \\ \frac{\partial X_t}{\partial \dot{x}_0} &= (1 / \omega) \sin \omega t \\ \frac{d}{dt} \frac{\partial X_t}{\partial \dot{x}_0} &= \cos \omega t \\ \frac{d}{dt} \frac{\partial X_t}{\partial \dot{x}_0} - \frac{\partial X_t}{\partial x_0} &= 0 \end{aligned}$$

 j -hat component (rotation plane)

$$\begin{aligned} \frac{\partial Y_t}{\partial y_0} &= \cos \omega t \\ \frac{\partial Y_t}{\partial \dot{y}_0} &= (1 / \omega) \sin \omega t \\ \frac{d}{dt} \frac{\partial Y_t}{\partial \dot{y}_0} &= \cos \omega t \\ \frac{d}{dt} \frac{\partial Y_t}{\partial \dot{y}_0} - \frac{\partial Y_t}{\partial y_0} &= 0 \end{aligned}$$

 k -hat component (rotation axis)

$$\begin{aligned} \frac{\partial Z_t}{\partial z_0} &= 1 \\ \frac{\partial Z_t}{\partial \dot{z}_0} &= 0 \\ \frac{d}{dt} \frac{\partial Z_t}{\partial \dot{z}_0} &= 0 \\ \frac{d}{dt} \frac{\partial Z_t}{\partial \dot{z}_0} - \frac{\partial Z_t}{\partial z_0} &= -1 \end{aligned}$$

which are zero for the X and Y components and -1 for the Z component. It has hence been demonstrated that the time dependent k -hat basis exponential (time dependent complex exponential) satisfies the Lagrange equation on the x - y plane of rotation and is hence the differential equation of a rotating reference frame

$$\frac{d}{dt} \frac{\partial X_t}{\partial \dot{x}_0} - \frac{\partial X_t}{\partial x_0} = 0 \quad \frac{d}{dt} \frac{\partial Y_t}{\partial \dot{y}_0} - \frac{\partial Y_t}{\partial y_0} = 0$$

On the axis of the rotation on the other hand

$$\frac{d}{dt} \frac{\partial Z_t}{\partial \dot{z}_0} - \frac{\partial Z_t}{\partial z_0} = -1$$

This, then must be an equation rather than an identity so the k -hat component which is neg-one is not a contradiction. It would be otherwise. I think it is the rotational eigenvalue since it corresponds to the **cosk** for the k -hat component of the rotational axis. Looking up the theory of Lagrangians, this suggests that this is a generalized force or something. Maybe it is a force that keeps the rotation latched to the k -hat axis or, say, a cosine of the angle that shows that the rotation axis is orthogonal to the plane of rotation [i.e., shows that the rotation axis (z) is the orthogonal complement of the plane of rotation].

When working in vector format which will be considered shortly, the neg-one for the Lagrange equation of the rotation axis could possibly be put to use as a means to determine the axis of rotation. If one knows that a variable is a basis exponential (a circular motion) but the orientation in space of the basis exponential is unknown and the Lagrangian is calculated, then for example if in MAPLE one could adjust the orientation until one obtains two zeroes and a negative-one similarly as adjusting a voltage or current to compare with known Lissajous figures to determine frequency.

As a divergence-like equation

$$\frac{d}{dt} \begin{pmatrix} \frac{\partial}{\partial \dot{x}_0} & \frac{\partial}{\partial \dot{y}_0} & \frac{\partial}{\partial \dot{z}_0} \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} - \begin{pmatrix} \frac{\partial}{\partial x_0} & \frac{\partial}{\partial y_0} & \frac{\partial}{\partial z_0} \end{pmatrix} \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = -1$$

where it is possible to put the second term in terms of nabla for a vector form of the Lagrange equation but not the first term because the partial derivatives are in terms of initial velocities. Maybe there should be some updated form of nabla for initial velocities.

Alternatively, noting the equations that comprise the components of the rotation matrix that diadically expand to the rotation matrix columns are

$$X_t \hat{i} = x_0 (\hat{i} \cos \omega t - \hat{j} \sin \omega t) \quad Y_t \hat{j} = y_0 (\hat{j} \cos \omega t + \hat{i} \sin \omega t) \quad Z_t \hat{k} = \hat{k} z_0$$

which are vector equations. These are the basis vectors of the cylindrical coordinate system (in the opposite sense [cw] because the exponent of the basis exponential is negative). Further noting with the trick $\hat{j} x_0 = \hat{i} (-\dot{x}_0 / \omega)$ and $\hat{i} y_0 = \hat{j} (\dot{y}_0 / \omega)$ that converts these equations of cylindrical basis vectors into the scalar equations

$$X_t = x_0 \cos \omega t + (\dot{x}_0 / \omega) \sin \omega t \quad Y_t = y_0 \cos \omega t + (\dot{y}_0 / \omega) \sin \omega t \quad Z_t = z_0$$

means that the time dependent k -hat **basis exponential** (x - y plane **rotation matrix**) can be written as

$$\begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} x_0 & \dot{x}_0 / \omega & 0 \\ y_0 & \dot{y}_0 / \omega & 0 \\ 0 & 0 & z_0 \end{pmatrix} \begin{pmatrix} \cos \omega t \\ \sin \omega t \\ 0 \end{pmatrix}$$

that is, as an expansion of basis vectors

$$\begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} \cos \omega t + \begin{pmatrix} \dot{x}_0 / \omega \\ \dot{y}_0 / \omega \\ 0 \end{pmatrix} \sin \omega t + \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix} \quad (\text{another way of expressing:}) \quad \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

which emphasizes that it is a solution to an eigenvalue problem, or in other words, the solution to a differential equation involving basis exponentials. The Lagrangian with respect to each of the backward variables then alternatively demonstrate the results in vector format

$$\begin{aligned} \left(\frac{d}{dt} \frac{\partial}{\partial \dot{x}_0} - \frac{\partial}{\partial x_0} \right) \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} &= \left(\frac{d}{dt} \frac{\partial}{\partial \dot{x}_0} - \frac{\partial}{\partial x_0} \right) \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} \cos \omega t + \left(\frac{d}{dt} \frac{\partial}{\partial \dot{x}_0} - \frac{\partial}{\partial x_0} \right) \begin{pmatrix} \dot{x}_0 / \omega \\ \dot{y}_0 / \omega \\ 0 \end{pmatrix} \sin \omega t + \left(\frac{d}{dt} \frac{\partial}{\partial \dot{x}_0} - \frac{\partial}{\partial x_0} \right) \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix} \\ &= -\frac{\partial}{\partial x_0} \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} \cos \omega t + \frac{d}{dt} \frac{\partial}{\partial \dot{x}_0} \begin{pmatrix} \dot{x}_0 / \omega \\ \dot{y}_0 / \omega \\ 0 \end{pmatrix} \sin \omega t + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos \omega t + \frac{d}{dt} \begin{pmatrix} 1 / \omega \\ 0 \\ 0 \end{pmatrix} \sin \omega t = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \cos \omega t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cos \omega t \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

and similarly

$$\begin{aligned} \left(\frac{d}{dt} \frac{\partial}{\partial \dot{y}_0} - \frac{\partial}{\partial y_0} \right) \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} &= \left(\frac{d}{dt} \frac{\partial}{\partial \dot{y}_0} - \frac{\partial}{\partial y_0} \right) \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} \cos \omega t + \left(\frac{d}{dt} \frac{\partial}{\partial \dot{y}_0} - \frac{\partial}{\partial y_0} \right) \begin{pmatrix} \dot{x}_0 / \omega \\ \dot{y}_0 / \omega \\ 0 \end{pmatrix} \sin \omega t + \left(\frac{d}{dt} \frac{\partial}{\partial \dot{y}_0} - \frac{\partial}{\partial y_0} \right) \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix} \\ &= -\frac{\partial}{\partial y_0} \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} \cos \omega t + \frac{d}{dt} \frac{\partial}{\partial \dot{y}_0} \begin{pmatrix} \dot{x}_0 / \omega \\ \dot{y}_0 / \omega \\ 0 \end{pmatrix} \sin \omega t + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos \omega t + \frac{d}{dt} \begin{pmatrix} 1 \\ 1/\omega \\ 0 \end{pmatrix} \sin \omega t = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \cos \omega t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos \omega t \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \left(\frac{d}{dt} \frac{\partial}{\partial \dot{z}_0} - \frac{\partial}{\partial z_0} \right) \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} &= \left(\frac{d}{dt} \frac{\partial}{\partial \dot{z}_0} - \frac{\partial}{\partial z_0} \right) \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} \cos \omega t + \left(\frac{d}{dt} \frac{\partial}{\partial \dot{z}_0} - \frac{\partial}{\partial z_0} \right) \begin{pmatrix} \dot{x}_0 / \omega \\ \dot{y}_0 / \omega \\ 0 \end{pmatrix} \sin \omega t + \left(\frac{d}{dt} \frac{\partial}{\partial \dot{z}_0} - \frac{\partial}{\partial z_0} \right) \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix} \\ &= -\frac{\partial}{\partial z_0} \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} \cos \omega t + \frac{d}{dt} \frac{\partial}{\partial \dot{z}_0} \begin{pmatrix} \dot{x}_0 / \omega \\ \dot{y}_0 / \omega \\ 0 \end{pmatrix} \sin \omega t - \frac{\partial}{\partial z_0} \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

that is, in summary for $\bar{R}_t = (X_t, Y_t, Z_t)' = e^{i\omega t \hat{k} \times} (x_0, y_0, z_0)'$, in vector format

$$\left(\frac{d}{dt} \frac{\partial}{\partial \dot{x}_0} - \frac{\partial}{\partial x_0} \right) \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \left(\frac{d}{dt} \frac{\partial}{\partial \dot{y}_0} - \frac{\partial}{\partial y_0} \right) \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \left(\frac{d}{dt} \frac{\partial}{\partial \dot{z}_0} - \frac{\partial}{\partial z_0} \right) \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

which contrasts to these equations when written as the scalar equations

$$\left(\frac{d}{dt} \frac{\partial}{\partial \dot{x}_0} - \frac{\partial}{\partial x_0} \right) X_t = 0 \quad \left(\frac{d}{dt} \frac{\partial}{\partial \dot{y}_0} - \frac{\partial}{\partial y_0} \right) Y_t = 0 \quad \left(\frac{d}{dt} \frac{\partial}{\partial \dot{z}_0} - \frac{\partial}{\partial z_0} \right) Z_t = -1$$

This is another demonstration of the true importance of Euler's formula (**expk = cisk**) now in the context of single component cross products (k -cross) as the complex number. Without Euler's formula, the true potential of k -cross as the complex number would not be possible. Similarly, without k -cross, Euler's formula is but a small significance compared to its real value and full potential, especially given that the complex part is no longer a **vestigial** part that gets thrown away [it is just as important as the real part, and taken together, are more valuable than each taken separately]. Euler's formula is already one of the most important formulas of mathematics so I wouldn't call it a contrivance, but compared to what it is now capable of [in the context of considering cross products as the complex numbers]. Since it is now known exactly what it is, it can be taken to arbitrarily large order (in $D = d/dt$ and overtones) for arbitrarily large orders of sophistication. [I considered calling Euler's formula a contrivance, despite knowing that it is not, simply (rather than out of arrogance) because it is non-calculus... it is instead probably **the main auxiliary equation of calculus** despite not itself being calculus].

Note that these are what I am stating are Lagrangians for electrons. I also have a really good solution for light (electromagnetic radiation) to improve solutions to Maxwell's solutions. This solution satisfies the Lagrangian equation for each of the principal directions [that is, the **Electric field component**, **Magnetic field component**, and the **Poynting vector... and so corresponds with Maxwell theory almost perfectly**]. Also of relevance is when wavelengths are large (small frequency) and the system can be considered DC (direct current) rather than AC (alternating current). In these circumstances the component sines and cosines reduce to the small angle formulae ($\cos \omega t = 1$ and $\sin \omega t = \omega t$ for small ω) so that the forward transition becomes

$$\begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} \dot{x}_0 \\ \dot{y}_0 \\ 0 \end{pmatrix} t$$

which is the position vector of a constant velocity (unit root) system on the x and y axes. This obviously also satisfies the Lagrangian.

$$\frac{d}{dt} \begin{pmatrix} \partial \dot{x}_0 / \partial \dot{x}_0 \\ \partial \dot{y}_0 / \partial \dot{y}_0 \\ \partial 0 / \partial \dot{z}_0 \end{pmatrix} t - \begin{pmatrix} \partial x_0 / \partial z_0 \\ \partial y_0 / \partial y_0 \\ \partial z_0 / \partial z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Lagrange equations of the k -hat double-cross basis exponential (Lagrangian of time dependent hyperbolics)

Lagrange's equation applies to harmonics and unit roots, as well as hyperbolics. Therefore, in addition to the standard basis exponentials that comprise rotation matrices which are harmonic [proved by Cauchy and Laplace equations], the Lagrangian also applies to the double crosses basis exponentials which are hyperbolic.

The harmonics of standard basis exponentials have been discussed to his point, that for the rotation matrix. Here I do the maths here for the hyperbolics, that is, for the k -hat double-cross basis exponential. This shows that the standard time dependent exponential (from regular maths) also satisfies Lagrange's equation when it is a solution to a differential equation (the trick being the backward variables which makes the solution appear to be the solution to a second order differential equation [if it is right, that is, if I haven't screwed up... time will tell]).

The **minimum polynomial** of the k -hat exponential [and **transfer function**] is $p(D) = D - \omega \hat{k} \times \hat{k} \times$. The unforced solution is the solution to $(D - \omega \hat{k} \times \hat{k} \times) \vec{R}_t = 0$. The unforced solution is

$$\vec{R}_t = e^{\omega t \hat{k} \times \hat{k} \times} \vec{r}_0 \quad \text{i.e.,} \quad (X_t, Y_t, Z_t)' = e^{\omega t \hat{k} \times \hat{k} \times} (x_0, y_0, z_0)'$$

$$\begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = e^{\omega t \hat{k} \times \hat{k} \times} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

where $e^{\omega t \hat{k} \times \hat{k} \times} = \mathbf{expkk}(\omega t)$. For each principal direction (the full mathematics is in the original k -cross and the k -hat exponential document)

$$i\text{-hat direction: } \mathbf{expkk}(\omega t) \hat{i} = \hat{i} \exp(-\omega t) = \hat{i} (\cosh(\omega t) - \sinh(\omega t))$$

$$j\text{-hat direction: } \mathbf{expkk}(\omega t) \hat{j} = \hat{j} \exp(-\omega t) = \hat{j} (\cosh(\omega t) - \sinh(\omega t))$$

$$k\text{-hat direction: } \mathbf{expkk}(\omega t) \hat{k} = \hat{k} (1) = \hat{k}$$

which do not cross over axes as for the single first order cross product. These diadically expand (as a cross-product operator) to the divergence/convergence matrix (a convergent star into a sink)

$$\mathbf{expkk}(\omega t) \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} e^{-\omega t} & 0 & 0 \\ 0 & e^{-\omega t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

which, since there is no cross-over, only have terms along the main diagonal. This could alternatively be obtained by Euler's formula for hyperbolics ($\mathbf{expkk} = \mathbf{ciskk}$)

$$\vec{R}_t = \vec{r}_0 (\mathbf{coskk}(\omega t) - \mathbf{sinkk}(\omega t) \hat{k} \times \hat{k} \times) \quad (X_t, Y_t, Z_t)' = (x_0, y_0, z_0)' (\mathbf{coskk}(\omega t) - \mathbf{sinkk}(\omega t) \hat{k} \times \hat{k} \times)$$

i -hat component (divergence plane)

$$\begin{aligned} X_t \hat{i} &= x_0 e^{\omega t \hat{k} \times \hat{k} \times} \hat{i} \\ &= x_0 (\mathbf{coskk} \omega t + (\mathbf{sinkk} \omega t) \hat{k} \times \hat{k} \times) \hat{i} \\ &= x_0 ((\mathbf{coskk} \omega t) \hat{i} + (\mathbf{sinkk} \omega t) \hat{k} \times \hat{k} \times \hat{i}) \\ &= x_0 ((\mathbf{coskk} \omega t) \hat{i} - (\mathbf{sinkk} \omega t) \hat{i}) \\ &= x_0 (\hat{i} (\cosh \omega t) - \hat{i} (\sinh \omega t)) \\ &= \hat{i} x_0 (\cosh \omega t - \sinh \omega t) \end{aligned}$$

j -hat component (divergence plane)

$$\begin{aligned} Y_t \hat{j} &= y_0 e^{\omega t \hat{k} \times \hat{k} \times} \hat{j} \\ &= y_0 (\mathbf{coskk} \omega t + (\mathbf{sinkk} \omega t) \hat{k} \times \hat{k} \times) \hat{j} \\ &= y_0 ((\mathbf{coskk} \omega t) \hat{j} + (\mathbf{sinkk} \omega t) \hat{k} \times \hat{k} \times \hat{j}) \\ &= y_0 ((\mathbf{coskk} \omega t) \hat{j} - (\mathbf{sinkk} \omega t) \hat{j}) \\ &= y_0 (\hat{j} (\cosh \omega t) - \hat{j} (\sinh \omega t)) \\ &= \hat{j} y_0 (\cosh \omega t - \sinh \omega t) \end{aligned}$$

k -hat component (principal axis)

$$\begin{aligned} Z_t \hat{k} &= z_0 e^{\omega t \hat{k} \times \hat{k} \times} \hat{k} \\ &= z_0 (\mathbf{coskk} \omega t + (\mathbf{sinkk} \omega t) \hat{k} \times \hat{k} \times) \hat{k} \\ &= z_0 ((\mathbf{coskk} \omega t) \hat{k} + (\mathbf{sinkk} \omega t) \hat{k} \times \hat{k} \times \hat{k}) \\ &= z_0 ((\mathbf{coskk} \omega t) \hat{k} + (\mathbf{sinkk} \omega t) (0) \hat{k}) \\ &= z_0 (\hat{k} (1) + \hat{k} (0)) \\ &= \hat{k} z_0 \end{aligned}$$

Hence the solution of the **principal directions** are

$$X_t \hat{i} = \hat{i} x_0 (\cosh \omega t - \sinh \omega t)$$

$$Y_t \hat{j} = \hat{j} y_0 (\cosh \omega t - \sinh \omega t)$$

$$Z_t \hat{k} = \hat{k} z_0$$

noting that these are scalar equations but do not have the right backward variables to demonstrate that the solutions satisfy the Lagrange equation. The differential equations of each of the three components are

$$DX_t - \omega X_t \hat{k} \times \hat{k} = 0$$

$$DY_t - \omega Y_t \hat{k} \times \hat{k} = 0$$

$$DZ_t - \omega Z_t \hat{k} \times \hat{k} = 0$$

For the second approach, taking directions immediately

in the **i -hat direction**

in the **j -hat direction**

in the **k -hat direction**

$$(D - \omega \hat{k} \times \hat{k}) X_t \hat{i} = 0$$

$$(D - \omega \hat{k} \times \hat{k}) Y_t \hat{j} = 0$$

$$(D - \omega \hat{k} \times \hat{k}) Z_t \hat{k} = 0$$

$$DX_t \hat{i} - \omega X_t \hat{k} \times \hat{k} \hat{i} = 0$$

$$DY_t \hat{j} - \omega Y_t \hat{k} \times \hat{k} \hat{j} = 0$$

$$DZ_t \hat{k} - \omega Z_t \hat{k} \times \hat{k} \hat{k} = 0$$

$$DX_t \hat{i} + \omega X_t \hat{i} = 0$$

$$DY_t \hat{j} + \omega Y_t \hat{j} = 0$$

$$DZ_t \hat{k} + 0 \hat{k} = 0$$

$$\hat{i} DX_t + \hat{i} \omega X_t = 0$$

$$\hat{j} DY_t + \hat{j} \omega Y_t = 0$$

$$\hat{k} DZ_t = 0$$

$$\hat{i} \dot{x}_0 + \hat{i} \omega x_0 = 0$$

$$\hat{j} \dot{y}_0 + \hat{j} \omega y_0 = 0$$

$$\hat{k} \dot{z}_0 = 0$$

i -hat component (divergence plane)

j -hat component (divergence plane)

k -hat component (principal axis)

$$\hat{i} \dot{x}_0 = \hat{i} (-\omega x_0) \Leftrightarrow \hat{i} x_0 = \hat{i} (-\dot{x}_0 / \omega)$$

$$\hat{j} \dot{y}_0 = \hat{j} (-\omega y_0) \Leftrightarrow \hat{j} y_0 = \hat{j} (-\dot{y}_0 / \omega)$$

$$\hat{k} \dot{z}_0 = 0$$

hence the somewhat trivial (because they are hyperbolic, and hence proportional, rather than hyperbolic so do not cross-over to alternate axes) but necessary substitutions $\hat{i} x_0 = \hat{i} (-\dot{x}_0 / \omega)$ and $\hat{j} y_0 = \hat{j} (-\dot{y}_0 / \omega)$, the component solutions are the same as those for the basis exponential rotation matrix but for hyperbolics (cosh and sinh) rather than harmonics (cos and sin)

i -hat component (divergence plane)

j -hat component (divergence plane)

k -hat component (principal axis)

$$X_t \hat{i} = x_0 e^{\omega t \hat{k} \times \hat{k}} \hat{i}$$

$$Y_t \hat{j} = y_0 e^{\omega t \hat{k} \times \hat{k}} \hat{j}$$

$$Z_t \hat{k} = z_0 e^{\omega t \hat{k} \times \hat{k}} \hat{k}$$

$$= \hat{i} (x_0 \cosh \omega t + (\dot{x}_0 / \omega) \sinh \omega t)$$

$$= \hat{j} (y_0 \cosh \omega t + (\dot{y}_0 / \omega) \sinh \omega t)$$

$$= \hat{k} z_0$$

which are still scalar equations for each component but the substitutions on the backward variables ($\hat{i} x_0 = \hat{i} (-\dot{x}_0 / \omega)$ and $\hat{j} y_0 = \hat{j} (-\dot{y}_0 / \omega)$) have converted the equations into essentially those of the harmonic solution but for cosh and sinh rather than sinh and cosh. These are now demonstrated to satisfy the Lagrange equation in the same way as for harmonics, that is, satisfy the equation for the orthogonal complement to the principal k -hat axis, that is for X and Y . For the principal axis, it is similarly negative one.

i -hat component (divergence plane)

j -hat component (divergence plane)

k -hat component (principal axis)

$$X_t = x_0 \cosh \omega t + (\dot{x}_0 / \omega) \sinh \omega t$$

$$Y_t = y_0 \cosh \omega t + (\dot{y}_0 / \omega) \sinh \omega t$$

$$Z_t = z_0$$

and in the form which have all of the variables required for Lagrange equation (**backward variables** and **time variable**). Now demonstrating that the components of the k -hat basis exponential satisfy Lagrange's equation on the plane of divergence (or convergence)

i -hat component (divergence plane)

j -hat component (divergence plane)

k -hat component (principal axis)

$$\frac{\partial X_t}{\partial x_0} = \cosh \omega t$$

$$\frac{\partial Y_t}{\partial y_0} = \cosh \omega t$$

$$\frac{\partial Z_t}{\partial z_0} = 1$$

$$\frac{\partial X_t}{\partial \dot{x}_0} = (1 / \omega) \sinh \omega t$$

$$\frac{\partial Y_t}{\partial \dot{y}_0} = (1 / \omega) \sinh \omega t$$

$$\frac{\partial Z_t}{\partial \dot{z}_0} = 0$$

$$\frac{d}{dt} \frac{\partial X_t}{\partial \dot{x}_0} = \cosh \omega t$$

$$\frac{d}{dt} \frac{\partial Y_t}{\partial \dot{y}_0} = \cosh \omega t$$

$$\frac{d}{dt} \frac{\partial Z_t}{\partial \dot{z}_0} = 0$$

$$\frac{d}{dt} \frac{\partial X_t}{\partial \dot{x}_0} - \frac{\partial X_t}{\partial x_0} = 0$$

$$\frac{d}{dt} \frac{\partial Y_t}{\partial \dot{y}_0} - \frac{\partial Y_t}{\partial y_0} = 0$$

$$\frac{d}{dt} \frac{\partial Z_t}{\partial \dot{z}_0} - \frac{\partial Z_t}{\partial z_0} = -1$$

which are the same solution as for the harmonic basis-exponential, zero for the X and Y components and -1 for the Z component. It has hence been demonstrated that the time dependent k -hat double cross basis exponential (time dependent regular exponential)

satisfies the Lagrange equation on the x - y plane of divergence/convergence and is hence the differential equation of a divergent or convergent reference frame

$$\frac{d}{dt} \frac{\partial X_t}{\partial \dot{x}_0} - \frac{\partial X_t}{\partial x_0} = 0 \quad \frac{d}{dt} \frac{\partial Y_t}{\partial \dot{y}_0} - \frac{\partial Y_t}{\partial y_0} = 0$$

and on the principal axis from which the double cross basis exponential diverges/converges

$$\frac{d}{dt} \frac{\partial Z_t}{\partial \dot{z}_0} - \frac{\partial Z_t}{\partial z_0} = -1$$

Similarly as for the harmonic basis exponential, the hyperbolic basis exponential can be put in the format which emphasizes that it is a solution to an eigenvalue problem. With the component solutions

$$X_t = x_0 \cosh \omega t + (\dot{x}_0 / \omega) \sinh \omega t \quad Y_t = y_0 \cosh \omega t + (\dot{y}_0 / \omega) \sinh \omega t \quad Z_t = z_0$$

means that the k -hat basis exponential (x - y plane rotation matrix) can be written exactly as the harmonic basis exponential (the rotation matrix) with the harmonics ($\cos$ and $\sin$) replaced by the hyperbolics ($\cosh$ and $\sinh$)

$$\begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} x_0 & \dot{x}_0 / \omega & 0 \\ y_0 & \dot{y}_0 / \omega & 0 \\ 0 & 0 & z_0 \end{pmatrix} \begin{pmatrix} \cosh \omega t \\ \sinh \omega t \\ 0 \end{pmatrix}$$

As an expansion of basis vectors

$$\begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} \cosh \omega t + \begin{pmatrix} \dot{x}_0 / \omega \\ \dot{y}_0 / \omega \\ 0 \end{pmatrix} \sinh \omega t + \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix} \quad (\text{another way of expressing:}) \quad \begin{pmatrix} X_t \\ Y_t \\ Z_t \end{pmatrix} = \begin{pmatrix} e^{-\omega t} & 0 & 0 \\ 0 & e^{-\omega t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

although explicitly recognises the time dynamics which is interesting. All of the solutions of the Lagrangian in vector format are the same as those for harmonics. For this reason, and for the sake of page count (as this was just meant to be a short note), these are omitted.

Note as one last word that the functions discussed here that have been proved to satisfy the Lagrange equation, those of the **basis exponentials**, are the functions that are observed in **bubble chambers** and **particle accelerators**. As I discussed previously, I think that it will also be bridging the gap between **circuit theory** (of applied electronics) and **field theory** (of electromagnetism) [with **quaternion impedance** Z and **quaternion admittance** Y which I defined in a previous document]. I think that this mathematics is going to become the mathematics of choice for **particle accelerator science**. Given that governments and consortiums are spending billions to trillions on the things, they might as well have the best mathematics (which in my opinion is this... calculus and the almighty cross product).